Vortices in the Jackiw-Pi model on the lattice

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Chern-Simons theory

For a general gauge field A the Chern-Simons action is

$$S_{\text{cs}} = -\frac{1}{2}\int \mathrm{tr}\left[A \wedge dA - \frac{2}{3} A \wedge A \wedge A\right]$$

- 3 (2+1) dimensional space-time
- Invariant under gauge transformations up to total derivatives
- No metric in the definition of the action (topological)

Schrödinger non-linear model

The action of the Schrödinger model

$$S_\psi = \int dtd^2x \psi^* i\partial_t \psi - \frac{1}{2}|\partial_i \psi|^2 - \frac{1}{2}g^2|\psi|^4$$

- Invariant under space-time translations
- Invariant under space rotations
- Invariant under Gallilean boosts (includes phase factor for ψ)
- Invariant under scale transformation ($t \to \lambda^2 t, x \to \lambda x, \psi \to \lambda^{-1}\psi$)
- Invariant under special conformal transformation
- Invariant under global $U(1)$ phase transformation

Jackiw-Pi model

$$S = S_{cs} + \int dt d^2 x \psi^* i D_t \psi - \frac{1}{2}|D_i \psi|^2 - \frac{1}{2} g^2 |\psi|^4$$
$$D_t = \partial_t - ieA_t$$
$$D_i = \partial_i - ieA_i$$

- Because S_{cs} is purely topological it has all the symmetries of the Schrödinger model
- The global $U(1)$ symmetry is now a local $U(1)$ gauge symmetry

Quantum mechanics of the Jackiw-Pi model

- Due to renormalization the coupling constants are now running.
- At 1-loop e does not renormalize
- $\beta(g_R^2) = \frac{1}{2\pi}\left(g_R^4 - e^4\right)$
- Thus at the special point $g_R^2 = \pm\frac{e^2}{m}$ conformal invariance is preserved

Equations of motion

- e.o.m. scalar potential $B = e\rho = e|\psi|^2$
- e.o.m. vector potential $E_i = e\epsilon_{ij}J_j$
- Schrödinger equation $i\partial_t\psi = \frac{-1}{2}D^2\psi - e\phi\psi - g^2|\psi|^2\psi$
- $\rho = |\psi|^2$
- $J = \frac{1}{2i}(\psi^* D\psi - (D\psi)^*\psi)$

Hamiltonian

The Hamiltonian of the system is given by

$$H = \int d^2x \frac{1}{2}|D\psi|^2 + \frac{g^2}{2}|\psi|^4$$

We introduce $D_\pm = \frac{1}{\sqrt{2}}(D_1 \pm iD_2)$ which satisfy

$$\frac{1}{2}D^2 = D_- D_+ - \frac{e}{2}B = D_+ D_- + \frac{e}{2}B$$

Hamiltonian

The Hamiltonian can thus be written

$$H = \int d^2x |D_\pm \psi|^2 + \frac{1}{2}(g^2 \pm e^2)|\psi|^4$$

Thus there exist stationary zero-energy solutions when

- $D_+\psi = 0$ and $g^2 + e^2 = 0$
- $D_-\psi = 0$ and $g^2 - e^2 = 0$

exactly when the theory is conformal QM at 1-loop

Vortex solutions

Solving $D_+\psi = 0$ leads to the Liouville equation

$$\Delta \log \sqrt{\rho} + e^2 \rho = 0$$

These have the general solution

$$\rho_f(z) = \frac{4}{e^2} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}$$

f is an meromorphic function (Horvathy and Yera [9805161]).
The poles of f are the centers of the vortices.

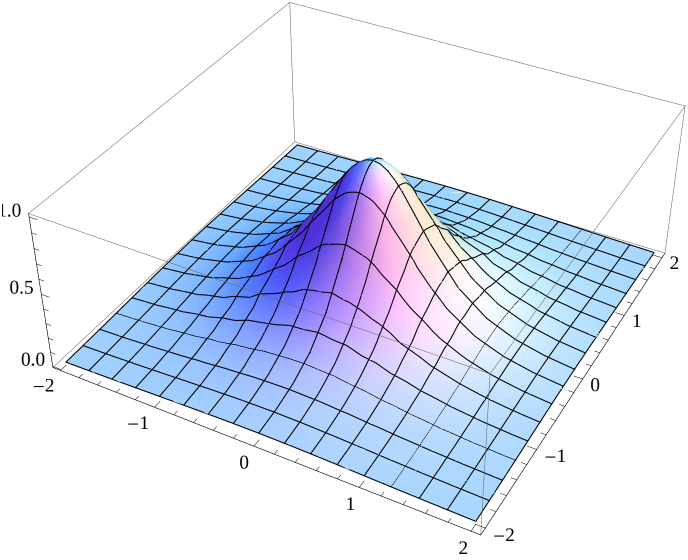
Clasification of the solutions on the plane

On the plane all solutions ρ are given by

$$f(z) = \frac{P(z)}{Q(z)}, \, \deg(P) < \deg(Q)$$

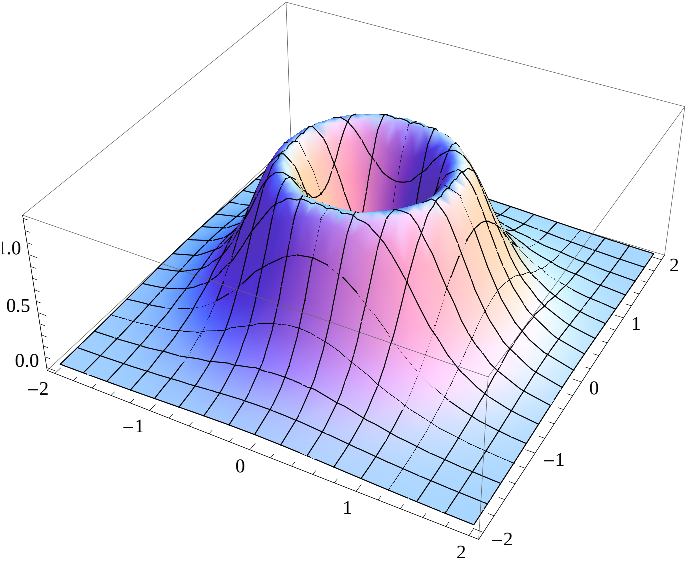
and vice versa (Horvathy and Yera [9805161]).

Plane examples 1



$$f = 1/z$$

Plane examples 2



$$f = 1/z^2$$

Total magnetic flux

$$\Phi = \int d^2x\rho = \int d^2x \frac{|f'|^2}{(1+|f|^2)^2} = \deg(Q)4\pi$$

Magnetic flux is quantized in units $q = \frac{\Phi}{2\pi} = 2\deg(Q)$

Solutions on the torus

- Impose periodic BC $\rho(z+\omega_i) = \rho(z)$
- This not equal to imposing periodic BC on ψ and A
- Naively: $\Phi = \int_T B = \int_{\partial T} A = 0$
- However we will see $q = \frac{\Phi}{2\pi} = n$

Classification of solutions on the torus

Periodic boundaries for ρ means that

$$\rho_f = \rho_g$$
$$g(z) = f(z + \omega)$$

Thus we need to classify which functions f, g lead to the same ρ

Classification of solutions on the torus

Insight: $\frac{|dz|}{1+|z|^2}$ is the metric of the riemann sphere.
Thus ρ is invariant under rotations $SO(3) = PSU(2)$ of the Riemann sphere.

$$g(z) = \frac{af(z) + b}{cf(z) + d} \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSU(2)$$

Theorem: if and only if

Classification of solutions on the torus

$$f(z+\omega_i) = U(\omega_i)f(z),\; U(\omega_i) \in PSU(2)$$
$$U(\omega_i)f(z) = \frac{af(z)+b}{cf(z)+d}$$

Therefor one has a homomorphism

$$\phi_f : \Lambda \to PSU(2)$$

Classification of solutions on the torus

Two options:

- $[\phi_f(\omega_1), \phi_f(\omega_2)] = 0$
  ϕ_f lift to a representation $\phi_f : \Lambda \to SU(2)$
- $\{\phi_f(\omega_1), \phi_f(\omega_2)\} = 0$
  ϕ_f lift to a representation $\phi_f : \Lambda^* \to SU(2)$

Commuting case

One can diagonalize the generators:

$$\phi_f(\omega_i) = \begin{pmatrix} e^{i\theta_i/2} & 0 \\ 0 & e^{-i\theta_i/2} \end{pmatrix}$$

Therefor one obtains

$$f(z+\omega_i) = e^{i\phi_i} f(z)$$

Elliptic functions of the second-kind.
A complete mathematical classifications exist

Anti-commuting case

Generators are

$$\phi_f(\omega_1) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$
$$\phi_f(\omega_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Therefor one obtains

$$f(z+\omega_1) = -f(z)$$
$$f(z+\omega_2) = 1/f(z)$$

Anti-commuting case

Suppose f_0 is a solution, then define

$$g(z) = f(z)/f_0(z)$$
$$h(z) = \frac{-g(z)+1}{g(z)+1}$$

Then

$$h(z+\omega_1) = h(z)$$
$$h(z+\omega_2) = -h(z)$$

h elliptic of the second kind with specific phase.

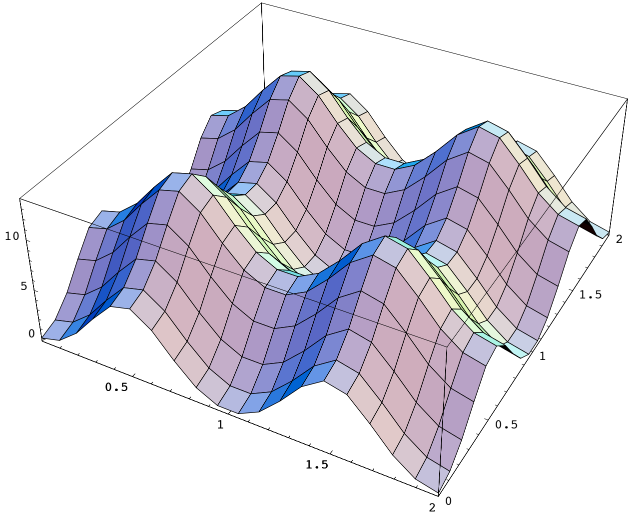
Anti-commuting case

For f_0 we use the Ansatz based on [Olesen Phys. Lett. B265 361-365]

$$f_0(z) = \frac{\mathcal{P}_{2\omega_1,2\omega_2} + b}{c\mathcal{P}_{2\omega_1,2\omega_2} + d}$$

where $\mathcal{P}_{2\omega_1,2\omega_2}$ is the Weierstrass p-function defined on a lattice with double periods.

Olesen solution



Olesen's solution (on rectangular grid)

Magnetic charge

- Commuting case: q=2n
- Anti-commuting case: q=2n+1

Like with spin one gets half-integer charge from the projective representation.

Summary and outlook

- A complete classification of all vortex solution on the torus has been given.
- May have some application in condensed matter systems.
- I'd like to understand the relation between the vortices and the vanishing of the conformal anomaly.
- Higher genus surfaces?